Sliding Mode Control for Turbulent Flows

Maxime Feingesicht, Andrey Polyakov, Franck Kerhervé, Jean-Pierre Richard

Sliding Mode Control for Turbulent Flows

Maxime Feingesicht *,**,*** **Andrey Polyakov** *,** **Franck Kerhervé** **** **Jean-Pierre Richard** *,**,***

* *Inria Lille Nord Europe, Parc scientifique de la Haute Borne, 40 Avenue du Halley Bat. A, Park Plaza, 59650 Villeneuve-d'Ascq, France*
** *CRIStAL UMR CNRS 9189, Université Lille 1, M3, Avenue Carl Gauss, 59650 Villeneuve-d'Ascq, France*
*** *Centrale Lille, Cité Scientifique, 59651 Villeneuve-d'Ascq, France*
**** *Institut P', CNRS UPR 3346, Université de Poitiers, 11 Boulevard Marie et Pierre Curie, BP 30179, F86962 Futuroscope Chasseneuil Cedex*

Abstract: The control of turbulent flows is a growing field with major interest, such as in aeronautical industries. This paper deals with the control of a flow separation. The dynamical model of the flow is identified based on experimental data using a specific identification procedure. A setpoint tracking problem is solved by means of Sliding Mode Control methodology. The theoretical results are supported with numerical simulations.

Keywords: Sliding Mode Control; Bilinear Systems; Time-Delay; Flow Control; Nonlinear Control

1. INTRODUCTION

Considering the modern technological challenges and the development of the transportation industry, the interest for control strategies applied to turbulent flows is constantly growing in the scientific and industrial communities (see Brunton and Noack (2015)). For example, it is commonly recognised that aerodynamic losses are the most important source of energy consumption for vehicules moving at high speeds (typically more than 50km/h). Recent ecological studies show that reducing these losses will drastically decrease the quantity of CO_2 emitted every year. For many years, the preferred solution was to optimize the shape of the vehicules in order to reduce their drag. This optimization acts as a passive control but it has now come close to its limitations and is not sufficient anymore. Recently, the interest for active control strategies (see Brunton and Noack (2015), Chun and Sung (1996), Selby et al. (1992)) has grown as the parameters of the controllers can be varied on-line in order to maintain an optimal solution, thus minimizing aerodynamic losses.

Air blowers are the most encountered actuators in turbulent flow control applications, at least at the university level (see, e.g. Volino (2003), McManus et al. (1994), Eldredge and Bons (2004)). The two-dimensional flap (see Raibaudo et al. (2013), Raibaudo et al. (2014), Chabert et al. (2014b), Chabert et al. (2014a), Shaqarin et al. (2013)) constitutes one of the standard benchmark of separated flow control system, extensively studied in the literature (see Chun and Sung (1996), Selby et al. (1992), Cierpka et al. (2007), Ciobaca and Wild (2013) and Åkervik et al. (2008)), and will be considered in the present paper as a test case to apply recently developed methods of the Control Theory.

The main issue of designing control algorithm for turbulent flows is that the dynamics of the plant (here the flow system) are highly nonlinear. The behavior of the flow is governed by the Navier-Stokes equation which is a partial diffential equation (PDE, also known as a distributed parameter model). Implementing controllers and observers on such kind of infinite dimensional systems is a complex task because of the enormous computational power requirement which is often a limitation for real-time applications (see Wachsmuth (2006), Ghattas and Bark (1997), Fernández-Cara et al. (2004)). Various strategies for separated flow control can be found in the literature. Most of them use very local (linear) models (i.e. basically skip nonlinear turbulent dynamics) and deal mainly with feedforward control (see Chun and Sung (1996), Selby et al. (1992), Cierpka et al. (2007), Ciobaca and Wild (2013) and Åkervik et al. (2008)). Recent development of control algorithm using machine learning (model-free) techniques (see, e.g. Duriez et al. (2014)) look promising as they are generic methods and can be applied to any kind of system. However, these methods require a very large number of experiments for proper tuning before being efficient. Furthermore, the robustness and convergence properties of the designed control are not totally proven and may cause unexpected behaviors. A recent survey about various approaches to flow control design is given by Brunton and Noack (2015). Model-based robust control of separated flows remains of particular interest and can be implemented in real systems without too much complexity if the model is well chosen, as well as having strong mathematical background and proofs. One of the objectives of the present paper is to study new perspectives in this topic.

As the control law is expected to be experimentally implemented, the model of the plant should be sufficiently simple. The model is chosen to be bilinear and presented as a difference-differential equation with state and input delays in order to capture the dynamics of the plant. It is to be noted that the control input is a relay ("ON"/"OFF") actuation provided by pulsed jets (air blowers). The model is identified using a modified 'gray-box' technique. The preliminary results on modeling of the control system of separated flows can be found in Feingesicht et al. (2016).

The present paper focuses on developing a setpoint tracking algorithm for the bilinear time-delay model obtained from experimental data of the flow. The control algorithm is designed using Sliding Mode methodology (see, e.g. Utkin (1992), Edwards and Spurgeon (1998), Shtessel et al. (2014)) jointly with a prediction technique (see, e.g. Fridman et al. (2001), Polyakov (2012)). The application of Sliding Mode for the considered system is justified as the actuators are designed for relay control (see Fridman et al. (2004), Yan et al. (2010)). Despite of the fact that bilinear systems were considered in literature (see, e.g. Gauthier and Kupka (1992)), to the best of our knowledge, the considered control problems for bilinear models with state and input delays has never been studied before.

Notation:

- $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$;
- $\mathbf{C}_\Omega$ is the space of continuous functions;
- $\mathbb{1} \in \mathbf{C}_\Omega$ is the unit constant function: $\mathbb{1}(s) = 1$, $\forall s \in \Omega$;
- $\mathbf{L}^2_\Omega$ is the space of quadratically integrable functions, $\|z\|_{\mathbf{L}^2_\Omega} = \sqrt{\int_\Omega z^2(s)ds}$;
- $\mathbf{L}^\infty_\Omega$ is the space of locally measurable essentially bounded functions, $\|z\|_{L^\infty} = \operatorname{ess\,sup}_{s\in\Omega} |z(s)|$;
- if $\tau > 0$, $y \in \mathbf{L}^\infty_\mathbb{R}$ and $t \in \mathbb{R}$ then $\xi_\tau(t) \in \mathbf{L}^\infty_{[-\tau,0]}$: $(y_\tau(t))(\sigma) = y(t+\sigma)$ for $\sigma \in [-\tau, 0]$. The notation $y_\tau(t)$ and $y_{-\tau}(t)$ is commonly used for time-delay models, as in Fridman (2014).

2. FLOW CONTROL SYSTEM

2.1 Flow Control Problem

The problem of flow control is the meeting point of several research areas (see Brunton and Noack (2015)) :

(1) Fluid Mechanics (for analysis of flow dynamics),
(2) Electronics (for sensing and actuation developments),
(3) Control Theory and Optimization (for formulation of control goals and designing of control laws).

Flow control experimental setup are generally designed and assembled based on current technological achievements in the field of fluid dynamics and electrical engineering. In such a context, the operator cannot have any impact on the set-up, except on the control parameters which drive the actuators. The problem resulting is therefore to optimize efficiency and robustness of the controller by designing appropriate control algorithms.

To the best of our knowledge, the paper presents the first attempts in the context of non-linear (in particular, bilinear) SISO model-based control design for separated flows. It deals with the problem of closed-loop (feedback) control, using Sliding Mode methodology in order to design a robust feedback which tracks a given setpoint.

2.2 Experimental test case

The experimental test case considered is that of a turbulent boundary layer flow occurring separation along a two-dimensional ramp whose geometry and dimensions are illustrated in Fig. 1. Full details of the experiments, which were conducted in the large boundary-layer wind tunnel at Laboratoire de Mécanique de Lille (France) can be found in Raibaudo et al. (2015), Raibaudo (2015). The boundary layer flow first develops along a flat horizontal plate (floor of the wind-tunnel) before reaching a smooth convergent where it occurs acceleration. The flow continues to develop along a slightly inclined flat plate to recover a zero pressure streamwise gradient. This is followed by a flap along which the boundary layer occurs separation and reattaches further downstream to the floor of the wind-tunnel. This is illustrated in Fig. 2 where streamlines for the averaged natural flow are reported. Note that the flow comes from the left of the figure. The ramp geometry is shown as the thick black line. In the present configuration, the location where the flow separates from the wall is fixed and located at the edge between the inclined flat plate and the flap (chosen as origin of the coordinate system in figure Fig. 2 and 3). Just downstream of the edge, a shear layer forms and a recirculation region (reversed flow) appears along the flap due to flow separation. The border between positive and negative streamwise mean velocity is represented as the blue line. Below this blue line, the flow is, in average, reversed compared to the flow above the line. The flow in this separation region constitutes the physical system of interest and to control, the main objective of the control being to reduce the recirculation region.

An array of 22 co-rotating round jets, air blowers, aligned parallel to the flap edge is used as actuators. The control input u(t) is a relay ("on"/"off") signal sent to the actuators with a given frequency and duty cycle. An example of the averaged flow obtained when using continuous actuation (relay remains "ON") is illustrated in Fig. 3. Compared to the natural flow discussed previously and shown in Fig. 2, the region of reversed flow is drastically reduced and the flow is found to be almost fully attached to the bottom wall.

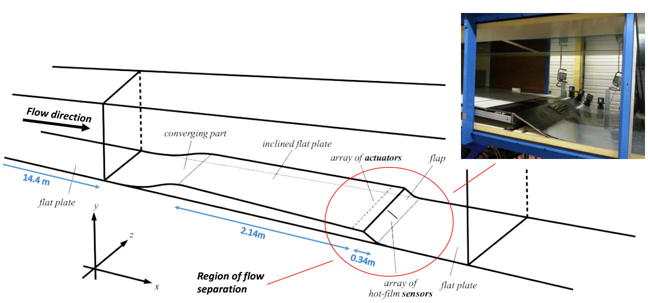

Fig. 1. Diagram and photo of the experimental setup Courtesy of Laboratoire de Mécanique de Lille

For real-time survey, hot-film sensors located along the flap are used to measure the gain in skin friction: an increase in friction gain being representative of flow reattachement. In the present configuration, output voltages of hot-film sensors are the only signals that can be measured in on-line and utilized for control proposes. The output voltages of the sensors are constants in the steady state. From the point of view of Control Theory, the control problem examined here admits conventional interpretation given in Fig. 4.

2.3 Control Aims : Setpoint Tracking Control

Based on flow velocity surveys (not detailed in the present paper), the capability of the actuators to reattach the flow to the wall has been shown by Raibaudo et al. (2015), Raibaudo (2015) to be well characterized by skin friction gain measured by the hot-film sensors. The control problem to be studied here is stabilization of the output y of the hot-films at the desired setpoint y^*. The relay nature of actuators motivates us to apply sliding mode principles

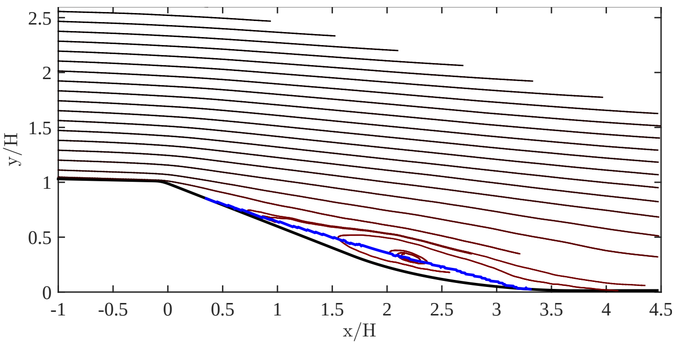

Fig. 2. Streamlines for the flow under continuous actuation

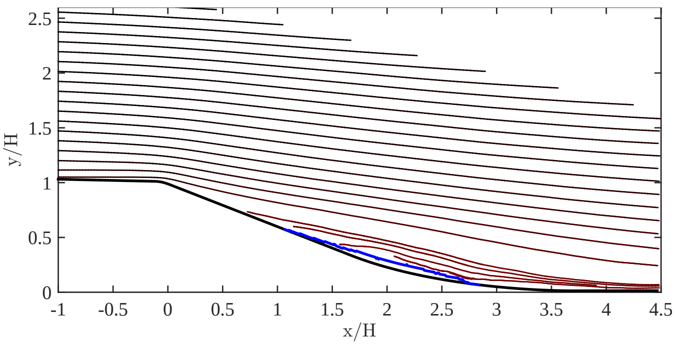

Fig. 3. Streamlines for the natural flow without control. The blue line represents the border between the reversed flow (negative streamwise velocity, region of the flow below the line) and the freestream (positive streamwise velocity, region of the flow above the line). In the controlled case Fig. 3 the recirculation region is shown to be drastically reduced and the flow almost fully reattached to the wall.

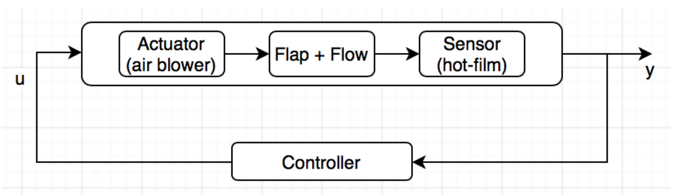

Fig. 4. Feedback scheme

in order to design a robust feedback law, which guarantees $y(t) \to y^*$ as $t \to \infty$.

3. INPUT-OUPUT MODELING OF SEPARATED FLOWS

3.1 Experimental Data and Pre-Processing

The only data we can use for modeling are the input signals to the actuators and the output voltages of the hot-film sensors measured with a frequency of $1kHz$. Therefore, we cannot design model separately for actuator, sensor and plant, but our model will implicitly include them all.

Several experiments have been done in order to collect an experimental database appropriate for model design. Each experiment consists of two phases: actuation and relaxation. Actuation is done by means of a periodic on/off input signal u with a fixed frequency and duty circle (DC). Actuation time is 5 seconds. Seven different input signals have been tested: 1) constant input; 2) Freq=4Hz with DC=50%; 3) Freq=4Hz with DC=80%; 4) Freq=8Hz with DC=50%; 5) Freq=8Hz with DC=80%; 6) Freq=80Hz with DC=50%; 6) Freq=80Hz with DC80%.

During the relaxation phase the control is switched off for 5 seconds in order to let the flow to return to a natural steady separated state. Each experiment is repeated for more than 50 times and the results are phase averaged in order to obtain an output signal less effected by measurement noises and exogenous perturbations. This phase-averaged data (see, Fig. 5) is utilized for modeling.

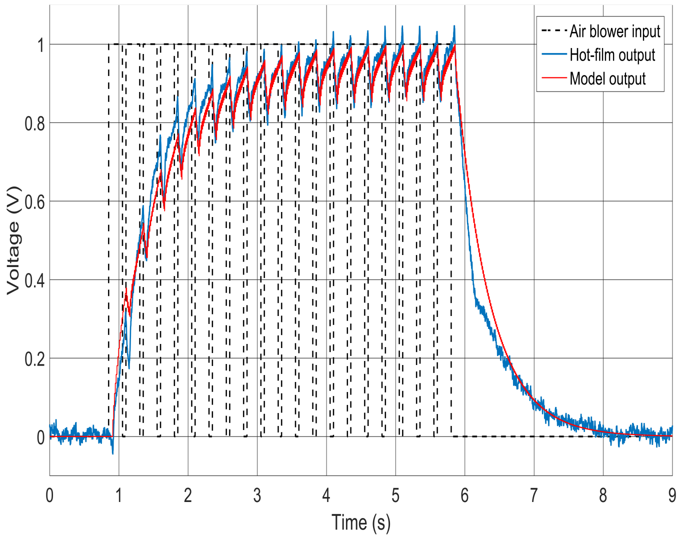

Fig. 5. Phase-averaged data for Freq=4Hz, DC=80%

3.2 Plant Model and Basic Assumptions

The dynamics of the flow considered here are highly non-linear and governed by partial differential equations (e.g. Navier-Stokes equations). Therefore, only a SISO (Single Input Single Output) model can be designed using the experimental dataset. However, this model should take into account nonlinearity and an infinite dimensional nature of the control system. That is why we identify an appropriate model from the class of bilinear control systems governed by differential equations with time delays (i.e. differential-difference equations).

For the design of a tracking control we use the following simple model (details on a more generic model can be found in Feingesicht et al. (2016)) :

$$\begin{aligned}\dot{y}(t) =& a_1 y(t-h) - a_2 y(t-\tau) \\ &+ (b - cy(t-h) + cy(t-\bar{\tau}))u(t-h),\end{aligned} \tag{1}$$

$$y(s) = 0, \quad u(s) = 0 \quad \text{for} \quad s \leq 0, \tag{2}$$

where $a_1 > 0, a_2 > 0, b > 0, c > 0, \bar{\tau} > 0, h > 0, \tau > 0$ are constant parameters, $y(t) \in \mathbb{R}$ - output and $u \in \mathbf{L}^\infty_{\mathbb{R}}$: $u(t) \in \{0,1\}, t \geq 0$ is the input. Note that for any $u \in L^\infty_{\mathbb{R}_+}$ the considered system has a unique Caratheodory solution Hale (1971) at least locally.

We deal with a model of physical system. To exclude non-feasible dynamics we assume that *the system* (1), (2) *has bounded positive solution for any input signal* $u \in \boldsymbol{L}^\infty_{\mathbb{R}_+}$: $u(t) \in \{0,1\}$. The sufficient condition of positivity and boundedness of solutions to the system (1) is given by the next proposition proven in Appendix.

Proposition 1. If $c < a_1$,$(a_1+c)\tau < a_2\tau < \frac{1}{e}$ *and* $\tau \leq h \leq \bar{\tau}$ *then the system* (1), (2) *is positive and its solution is globally bounded for any input signal* $u \in \boldsymbol{L}^\infty : u(t) \in \{0,1\}$ *as follows*

$$0 \leq y(t) < y_{\max} := \frac{b}{a_2 - a_1} \text{ for all } t \geq 0.$$

3.3 Model identification

The precision of the models has been analyzed using the three indicators : ϵ is L_2-norm of the error, FIT index introduced in Dandois et al. (2013) and ρ - the correlation between the experimental data and the identified model.

$$\epsilon = \|y_{exp} - y_{sim}\|_{\mathbf{L}^2}, \quad \rho = \frac{\operatorname{cov}(y_{exp}, y_{sim})}{\sigma_{y_{exp}}\sigma_{y_{sim}}},$$

$$\text{FIT} = \left(1 - \frac{\|y_{exp} - y_{sim}\|_{\mathbf{L}^2}}{\|y_{exp} - \bar{y}_{exp}\|_{\mathbf{L}^2}}\right) \times 100\%,$$

where y_{exp} is the output of the system obtained from the experiment, y_{sim} is the output generated by the identified bilinear model (1), $\bar{y}_{exp}$ is the mean value of y^{exp}, $\operatorname{cov}(y_{exp}, y_{sim})$ is the covariance of y_{exp} and y_{sim}, but $\sigma_{y_{exp}}$ and $\sigma_{y_{sim}}$ are standard deviations of y_{exp} and y_{sim}, respectively.

The model is identified using a least-square method supported with global optimization algorithm NOMAD (see, Audet et al. (2009); Le Digabel (2011); Audet et al. (2007); Audet and Dennis (2006)) required for optimal assignment of delays. The reader can refer to Feingesicht et al. (2016) for more details about identification of the considered bilinear model.

3.4 Identification results

The results of the identification are summarized in Tables 1 and 2.

It is worth stressing that the obtained models have very high precision comparing with the existing results Dandois et al. (2013). The FIT index is improved for almost 30% using a model with only 10 parameters (5 delays and 5 coefficients, see, Table I). The NARX model obtained in Dandois et al. (2013) has thousands of coefficients and FIT=59%. A plot of the identified model and experimental data can be found in Fig. 5.

Table 1. Identified parameters of the model

τ_i	$\left[0.054; 0.006\right]$
$\bar{\tau}_j$	$\left[0.054; 0.360\right]$
h_k	0.054
a_i	$\left[9.5122; -12.5188\right]$
b_k	3.5515
c_{jk}	$\left[-2.2430; 2.2430\right]$

Table 2. Precision of the identified model

ε	FIT	ρ
0.4498	87.11%	0.9918

4. SLIDING MODE CONTROL FOR SEPARATED FLOWS

The conventional sliding mode control methodology (see, e.g. Utkin (1992), Edwards and Spurgeon (1998), Shtessel et al. (2014)) is developed for delay-free systems. In order to design the sliding mode control we need to compensate input delay using, for example, the prediction technique (see, e.g. Olbrot (1976), Artstein (1982), Fridman et al. (2001), Polyakov (2012)). Developed originally for linear plants this idea can also be applied for bilinear systems under consideration. Assuming $\bar{\tau} > h$ we introduce the following sliding variable

$$\begin{aligned}\sigma(t) =& y(t) - a_2 \int_{t-\tau}^{t} y(s)ds + c\int_{t-\bar{\tau}+h}^{t} y(s)ds \\ &+ \int_{t-h}^{t} a_1 y(s) + (b - cy(s) + cy(s-\bar{\tau}+h))u(s)ds.\end{aligned} \tag{3}$$

Obviously, the variable σ satisfies the equation

$$\dot{\sigma}(t) = (a_1 - a_2 + c(1-u(t)))y(t) + c(u(t)-1)y(t-\bar{\tau}+h) + bu(t).$$

Note that the control input u is not delayed with respect to the sliding variable σ, so the conventional sliding mode existence conditions can be utilized (see, Utkin (1992)) for control design.

Proposition 2. If conditions of Proposition 1 hold and

$$Q(j\omega) \neq 0 \quad \text{for} \quad \omega \neq 0, \tag{4}$$

where $Q(s) = s + a_2 e^{-s\tau} - (a_2 - c)e^{-sh} - ce^{-s\bar{\tau}}, s \in \mathbb{C}$ *and* $j = \sqrt{-1}$, *then the control law*

$$u(t) = \begin{cases} 1 & \text{if } \sigma(t) < \sigma^*, \\ 0 & \text{if } \sigma(t) > \sigma^*, \end{cases} \tag{5}$$

with $\sigma^* = y^*(1 + a_2(h-\tau) + c(\bar{\tau} - h))$ *and* $y^* \in \left(0, \frac{b}{a_2 - a_1}\right)$ *guarantees* $y(t) \to y^*$ *as* $t \to +\infty$.

The proof of this proposition is given in Appendix, where it is shown that the control (5) guarantees finite-time convergence of the sliding variable $\sigma(t)$ to σ^*, so $\sigma(t) = \sigma^*$

for all $t \geq T$. It is worth stressing that when sliding mode arises the system motion is governed by the infinite dimensional dynamic system

$$\sigma^* = y(t) + a_2 \int\limits_{t-h}^{t-\tau} y(s)ds + c \int\limits_{t-\bar{\tau}}^{t-h} y(s)ds.$$

This means that our sliding surface $\sigma = \sigma^*$ is "inifine dimensional". Using condition (4) it is proven that any trajectory $y(t)$ of this system convergence y^* asymptotically as $t \to \infty$.

Remark 1. Since

$$Re(Q(j\omega)) = a_2 \cos(\tau\omega) - (a_2 - c)\cos(h\omega) - c\cos(\bar{\tau}\omega)$$
$$Im(Q(j\omega)) = \omega - a_2 \sin(\tau\omega) + (a_2 - c)\sin(h\omega) + c\sin(\bar{\tau}\omega)$$

then to check the condition (4) *it is sufficient to consider* $\omega \in (0, 2(a_2 + c)]$.

5. NUMERICAL SIMULATIONS OF SETPOINT TRACKING CONTROL

Obviously, the plant model obtained by the identification (see, Table I, $N_3 = 1$) satisfies the proposition 1 with $a_1 = 9.6468$, $a_2 = 12.6195$, $c = 2.6470$, $b = 3.5632$, $\tau = 0.006$, $h = 0.054$, $\bar{\tau} = 0.360$ and with the admissible setpoint value $y^* \in (0, y_{\max}), y_{\max} = \frac{b}{a_2 - a_1} = 1.20$. According to Remark 1 the condition (4) has been validated graphically using the parametric plot of the function Q in the complex plane (see, Fig. 6).

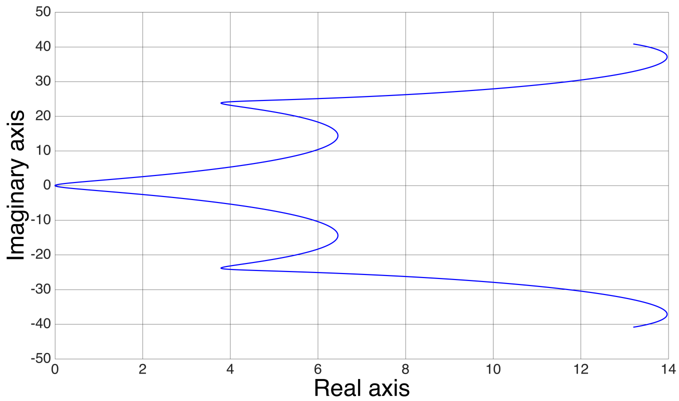

Fig. 6. Parametric plot for $Q(j\omega)$ for $-2(a_2 + c) < \omega < +2(a_2 + c)$

A numerical simulation result are depicted in Fig. 7 and Fig. 8 for $y^* = 0.85$ ($\sigma^* = 2.0534$, respectively). The simulation has been done using explicit Euler method and a rather large step size 10^{-3}.

Fig. 8 shows that after approximately 1s, the sliding mode appears making the input oscillate between 0 and 1 at high frequency. Next, the output $y(t)$ converges to the desired setpoint y^* with an error of the order 10^{-3}. The numerical simulations have been also made for the smaller step size 10^{-5}. They confirmed convergence of $y(t)$ to y^* with an error of the order 10^{-5}, which corresponds to numerical precision of the Euler method.

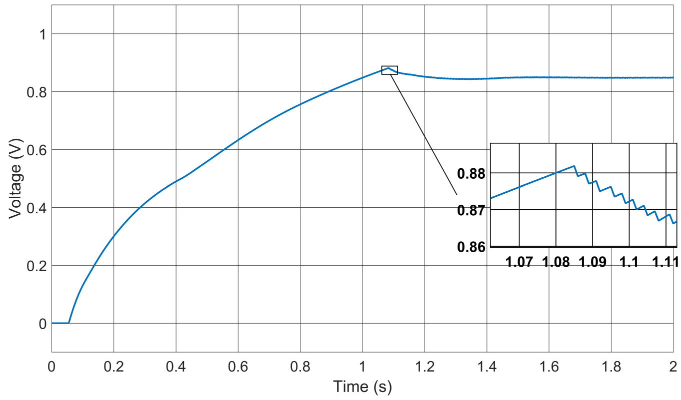

Fig. 7. Application of the setpoint tracking control: Output of the system

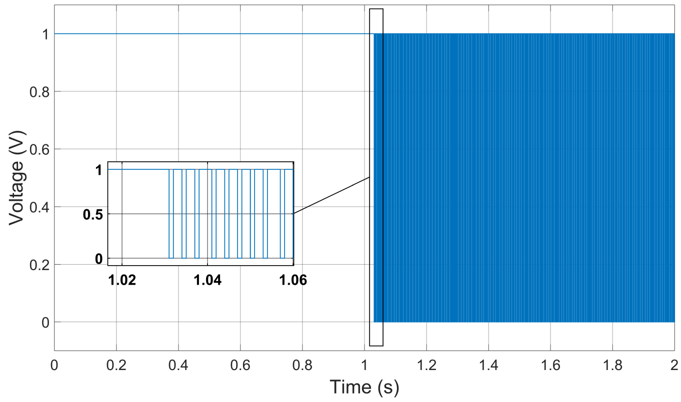

Fig. 8. Application of the setpoint tracking control: Control signal

6. DISCUSSION AND CONCLUSIONS

In the present paper, the problem of model-based closed loop control of separated flows has been studied. The model is defined as a bilinear time-delay system and identified based on experimental data. A nonlinear robust feedback is then developed, based on the Sliding Mode methodology and tested on the identified model. The experimental tests of the proposed control strategy as well as extensions to more exact models of separated flows are planned for future works.

7. APPENDIX

7.1 Positive systems with time-delay

Lemma 1. If $a > 0, \tau > 0$ *and* $a\tau < \frac{1}{e}$ *then the system* $\dot{z}(t) = -az(t-\tau) + f(t)$, $z(s) = 0$ *for* $s \leq 0$ *is positive for any non-negative* $f \in \boldsymbol{L}_{\mathbb{R}}^{\infty}$, *i.e.* $z(t) \geq 0$ *if* $f(t) \geq 0$ *for all* $t \geq 0$.

Proof. See Lemma 4 of Efimov et al. (2015) or Corollary 15.9 from Agarwal et al. (2012). ■

Lemma 2. Let the system $\dot{z}(t) = -az(t-\tau) + b, z(s) = 0$ *for* $s \leq 0$ *be positive and* $a > 0, b > 0$, $0 \leq a\tau < \ln(2)$. *Then it has a unique solution defined on* $\mathbb{R}_+$ *such that* $0 < z(t) < \frac{b}{a}$ *and* $\dot{z}(t) > 0$ *for all* $t > 0$.

Proof. Let us suppose a contrary, i.e. there exists $t^* > 0$ such that $z(t^*) = \frac{b}{a}$, but $z(t) < \frac{b}{a}$ for all $t > t^*$. This immediately implies that $\dot{z}(t) > 0$ and $z(t) > 0$ for all $t \in (0, t^*]$.

Let us denote $p(t) = z(t) - a\int_{t-\tau}^{t} z(s)ds$. Hence, we have

$$\dot{p}(t) = -az(t) + b = -ap(t) + b - a^2 \int\limits_{t-\tau}^{t} z(s)ds \text{ and}$$

$$z(t) = \tfrac{b(1-e^{-at})}{a} + a\int\limits_{t-\tau}^{t} z(s)ds - a^2\int\limits_{0}^{t} e^{-a(t-\sigma)} \int\limits_{\sigma-\tau}^{\sigma} z(s)dsd\sigma =$$
$$\tfrac{b(1-e^{-at})}{a} + a\int\limits_{t-\tau}^{t} z(s)ds - a\int\limits_{0}^{t} e^{-a(t-\sigma)} \int\limits_{t-\tau}^{t} z(\sigma - t + s)dsd\sigma =$$
$$\tfrac{b(1-e^{-at})}{a} + a\int\limits_{t-\tau}^{t}\left(z(s) - a\int\limits_{0}^{t} e^{-a(t-\sigma)} z(\sigma - t + s)d\sigma\right)ds =$$
$$\tfrac{b(1-e^{-at})}{a} + a\int\limits_{t-\tau}^{t}\left(z(s) - a\int\limits_{-t}^{0} e^{a\sigma} z(s+\sigma)d\sigma\right)ds =$$
$$\tfrac{b(1-e^{-at})}{a} + a\int\limits_{t-\tau}^{t}\left(z(s) - ae^{-as}\int\limits_{s-t}^{s} e^{a\sigma} z(\sigma)d\sigma\right)ds =$$
$$\tfrac{b}{a}(1 - e^{-at}) + a\int_{t-\tau}^{t} e^{-as} f(s,t)ds,$$

where $f(s,t) = e^{as}z(s) - a\int_{s-t}^{s} e^{a\sigma} z(\sigma)d\sigma$. Since for $s \in [t-\tau, t]$ and $0 < t \le t^*$ we have

$$\frac{\partial f}{\partial s} = e^{as}\dot{z}(s) + ae^{as}z(s) - ae^{as}z(s) + ae^{a(s-t)}z(s-t) \ge 0$$

then $z(t) \le \frac{b(1-e^{-at})}{a} + af(t,t)\int\limits_{t-\tau}^{t} \frac{ds}{e^{as}} = \frac{b(1-e^{-at})}{a} + f(t,t)\frac{e^{a\tau}-1}{e^{at}}$ or, equivalently,

$$z(t) \le \tfrac{b(1-e^{-at})}{a(2-e^{a\tau})} - \frac{a(e^{a\tau}-1)}{2-e^{a\tau}} \int_0^t e^{-a(t-\sigma)} z(\sigma)d\sigma.$$

Hence, $z(t) \le w(t)$, where $w(t)$ satisfies the integral equation $w(t) = \frac{b(1-e^{-at})}{a(2-e^{a\tau})} - \frac{a(e^{a\tau}-1)}{2-e^{a\tau}} \int_0^t e^{-a(t-\sigma)} w(\sigma)d\sigma, w(0) = 0$ or, equivalently,

$$\dot{w}(t) = -a\left(1 + \tfrac{e^{a\tau}-1}{2-e^{a\tau}}\right) w(t) + \tfrac{b}{2-e^{a\tau}} = \tfrac{-aw(t)+b}{2-e^{a\tau}}.$$

Therefore, we derive that $z(t) \le \frac{b}{a}\left(1 - e^{-\frac{a}{2-e^{a\tau}}t}\right) < \frac{b}{a}$ for all $t > 0$. This contradicts our supposition. ∎

7.2 Proof of Proposition 1

I. Let us consider the system $\dot{y}(t) = -a_2 y(t-\tau) + f(t),\ y(s) = 0, s \le 0$, where f is a locally integrable function. If $f(t) \ge 0$ and $a_2\tau \le e^{-1}$ then the delay-dependent positivity conditions hold (see, Lemma 1) and $y(t) \ge 0$, for all $t \ge 0$. On the one hand, if $a_1 \ge c$ and $y(s) \ge 0$ for $s \le t$ then $f(t) = a_1 y(t-h) + (b - cy(t-h) + cy(t-\bar{\tau}))u(t-h) \ge 0$. Therefore, using the method of steps (i.e. considering sequentially the intervals $[0,h], [h,2h]$...) we prove positivity of the system (1), (2).

II. Now let us prove boundedness of solutions. Suppose the contrary: there exists an input signal $u(t)$ and an instant of time $t^* > 0 : y(t^*) = y_{\max}$ and $y(s) < y_{\max}$ for $s < t^*$.

In this case, since $b - cy(t-h) + cy(t-\bar{\tau}) \ge b - cy(t-h) = (a_2 - a_1)y_{\max} - cy(t-h) > 0$ for all $t \in [0, t^*]$ then $y(s) \le y_{\mathbb{1}}(s)$ for all $s \le t^*$, where $y_{\mathbb{1}}$ is the solution to the positive system $\dot{y}_{\mathbb{1}}(t) = (a_1 - c)y_{\mathbb{1}}(t-h) - a_2 y_{\mathbb{1}}(t-\tau) + cy_{\mathbb{1}}(t-\bar{\tau}) + b$. Let us prove boundedness of solutions of the latter system for this purpose let us study the auxiliary system

$$\dot{z}(t) = -(a_2 - a_1)z(t-\tau) + b, \quad z(s) = 0 \text{ for } s \le 0.$$

For $\Delta(t) = z(t) - y_{\mathbb{1}}(t)$ we derive $\dot{\Delta}(t) = -a_2\Delta(t-\tau) + a_1 z(t-\tau) - (a_1 - c)y_{\mathbb{1}}(t-h) - cy_{\mathbb{1}}(t-\bar{\tau})$. According to Lemma 2 the function z satisfies the inequalities $0 < z(t) < b/(a_2 - a_1) = y_{\max}$ and $\dot{z}(t) > 0$ for all $t > 0$. Hence, $z(t-\tau) \ge z(t-h) \ge z(t-\bar{\tau})$ and

$$\dot{\Delta}(t) = -a_2\Delta(t-\tau) + (a_1 - c)\Delta(t-h) + c\Delta(t-\bar{\tau}) + \eta(t),$$

where $\eta(t) = a_1 z(t-\tau) - (a_1 - c)z(t-h) - cz(t-\bar{\tau}) \ge 0$ and $\Delta(s) = 0$ for $s \le 0$. Since the latter system is positive (see, the first part of this proof) then $\Delta(t) \ge 0$ and $y_{\mathbb{1}}(t) \le z(t) < y_{\max}$ for all $t \ge 0$. This contradicts our supposition.

7.3 Proof of Proposition 2

Firs of all, let us note that $0 < y(t) < \frac{b}{a_2 - a_1}$ for all $t > 0$ due to Proposition 1.

I. Since the system (1) is positive, then $y(t) \ge 0$ for all $t \ge 0$. Moreover, if $u(t) = 0$ and $y(t) > 0$ then $\dot{\sigma}(t) < 0$, but if $u(t) = 1$ and $y(t) < \frac{b}{a_2 - a_1}$ then $\dot{\sigma}(t) > 0$. Therefore, while $0 < y(t) < \frac{b}{a_2 - a_2}$ we have $(\sigma(t) - \sigma^*)\frac{d}{dt}(\sigma(t) - \sigma^*) < 0$. Obviously, $\sigma(0) = 0$. To guarantee existence of sliding mode we just need to show that the state $\sigma(t) = \sigma^* > 0$ is reachable in a finite time $t = t^* > 0$. Let us suppose contrary: $\sigma(t) < \sigma^*$ for all $t > 0$. This means that $u(t) = 1$ for all $t > 0$ and $\dot{y}(t) = (a_1 - c)y(t-h) - a_2 y(t-\tau) + cy(t-\bar{\tau}) + b$. Using the last identity let us rewrite the formula (3) as

$$\begin{aligned}\sigma(t) &= y(t) - a_2\int_{t-\tau}^{t} y(s)ds + c\int\limits_{t-\bar{\tau}+h}^{t} y(s)ds \\ &+ \int\limits_{t-h}^{t} \dot{y}(s+h) + a_2 y(s+h-\tau)ds \\ &= 2y(t) - y(t-h) + a_2\int_{t}^{t+h-\tau} y(s)ds \qquad (6) \\ &+ c\int\limits_{t-\bar{\tau}+h}^{t} y(s)ds.\end{aligned}$$

Hence, $\sigma(t) \ge \sigma^*$ if $y(t) \ge y^*$. Let us show that there exists $t^* > 0$ such that $y(t) > y^*$ for all $t > t^*$. Since $\dot{y}(t) > -a_2 y(t-\tau) + b$ for all $t > 0$ then, obviously, there exists $t_1 > 0$ such that $y(t) > \frac{b}{a_2}$ for all $t > t_1$. In this case, we derive $\dot{y}(t) > -a_2 y(t-\tau) + b\left(1 + \frac{a_1}{a_2}\right)$ for all $t > t_1 + \bar{\tau}$ and there exists $t_2 > t_1 + \bar{\tau}$ such that $y(t) > \frac{b}{a_2}\left(1 + \frac{a_1}{a_2}\right)$ for all $t > t_2$, and so on. Therefore, for $t > t_i$ we derive

$$y(t) > \tfrac{b}{a_2}\left(1 + \tfrac{a_1}{a_2} + \ldots + \left(\tfrac{a_1}{a_2}\right)^{i-1}\right) = \frac{b\left(1 - \left(\frac{a_1}{a_2}\right)^i\right)}{a_2 - a_1}$$

and for some i^* we have $y(t_{i^*}) > y^* \in \left(0, \frac{b}{a_2 - a_1}\right)$. Therefore, the sliding mode existence condition Utkin (1992) holds and $\sigma(t) = \sigma^*, \forall t > t^*$.

II. Using the equivalent control method Utkin (1992) we derive $u_{eq}(t) = \frac{(a_2 - a_1 - c)y(t) + cy(t-\bar{\tau}+h)}{b - cy(t) + cy(t-\bar{\tau}+h)}$ and

$$\sigma^* = y(t) - a_2 \int\limits_{t-\tau}^{t} y(s)ds + c \int\limits_{t-\bar{\tau}+h}^{t} y(s)ds$$
$$+ \int\limits_{t-h}^{t} a_2 y(s) - cy(s) + cy(s - \bar{\tau} + h)ds$$
$$= y(t) + a_2 \int\limits_{t-h}^{t-\tau} y(s)ds + c \int\limits_{t-\bar{\tau}}^{t-h} y(s)ds$$

for all $t > t^*$. Introducing the variable $\Delta(t) = y(t) - y^*$ we obtain the equation

$$\Delta(t) + a_2 \int\limits_{t-h}^{t-\tau} \Delta(s)ds + c \int\limits_{t-\bar{\tau}}^{t-h} \Delta(s)ds = 0. \tag{7}$$

It has the characteristic equation $\frac{1}{s}Q(s) = 0$, $s \in \mathbb{C}$. We have already proven that all solutions of the closed-loop system are bounded (see, Proposition 1) and the sliding mode exists for all $t > t^*$, so the equation (7) does not have unbounded dynamics. The condition $Q(j\omega) \neq 0$ for all $\omega \neq 0$ implies that this equation does not have non-constant periodic solutions. So, the only stable solution is $\Delta(t) \equiv C$, where $C \in \mathbb{R}$ is some constant. Since $1 + a_2(h - \tau) + c(\bar{\tau} - h) > 0$ then from the equation for $\Delta(t)$ we immediately derive $C = 0$ and $y(t) \to y^*$ as $t \to \infty$.

ACKNOWLEDGEMENTS

This work was supported by the Région Nord Pas-de-Calais in the framework of the ELSAT-CISIT program. The authors are indebted to C. Raibaudo for the courtesy of the experimental data and to the LML for the experimental campaign conducted.

REFERENCES

Agarwal, R.P., Berezansky, L., Braverman, E., and Domoshnitsky, A. (2012). *Nonoscillation Theory of Functional Differential Equations with Applications*. Springer New York, New York, NY.

Åkervik, E., Hoepffner, J., Ehrenstein, U., and Henningson, D.S. (2008). Model reduction and control of a cavity-driven separated boundary layer. In *IUTAM Symposium on Flow Control and MEMS*, 147–155. Springer.

Artstein, Z. (1982). Linear systems with delayed controls: A reduction. *IEEE Transactions on Automatic Control*, 27(4), 869–879. doi:10.1109/TAC.1982.1103023.

Audet, C. and Dennis, J. (2006). Mesh Adaptive Direct Search Algorithms for Constrained Optimization. *SIAM Journal on Optimization*, 17(1), 188–217. doi:10.1137/040603371.

Audet, C., Béchard, V., and Digabel, S.L. (2007). Nonsmooth optimization through Mesh Adaptive Direct Search and Variable Neighborhood Search. *Journal of Global Optimization*, 41(2), 299–318. doi:10.1007/s10898-007-9234-1.

Audet, C., Jr, J.E.D., and Digabel, S.L. (2009). Globalization strategies for Mesh Adaptive Direct Search. *Computational Optimization and Applications*, 46(2), 193–215. doi:10.1007/s10589-009-9266-1.

Brunton, S.L. and Noack, B.R. (2015). Closed-Loop Turbulence Control: Progress and Challenges. *Applied Mechanics Reviews*, 67(5), 050801–050801. doi:10.1115/1.4031175. URL `http://dx.doi.org/10.1115/1.4031175`.

Chabert, T., Dandois, J., and Garnier, É. (2014a). Experimental closed-loop control of flow separation over a plain flap using slope seeking. *Experiments in Fluids*, 55(8), 1–19.

Chabert, T., Dandois, J., Garnier, É., and Jacquin, L. (2014b). Experimental detection of flow separation over a plain flap by wall shear stress analysis with and without steady blowing. *Comptes Rendus Mécanique*, 342(6), 389–402.

Chun, K.B. and Sung, H.J. (1996). Control of turbulent separated flow over a backward-facing step by local forcing. *Experiments in Fluids*, 21(6), 417–426. doi:10.1007/BF00189044.

Cierpka, C., Weier, T., and Gerbeth, G. (2007). Electromagnetic Control of Separated Flows Using Periodic Excitation with Different Wave Forms. In P.D.R. King (ed.), *Active Flow Control*, number 95 in Notes on Numerical Fluid Mechanics and Multidisciplinary Design (NNFM), 27–41. Springer Berlin Heidelberg.

Ciobaca, V. and Wild, J. (2013). An Overview of Recent DLR Contributions on Active Flow-Separation Control Studies for High-Lift Configurations. *Aerospace Lab*, Issue(2013-06).

Dandois, J., Garnier, E., and Pamart, P.Y. (2013). NARX modelling of unsteady separation control. *Experiments in fluids*, 54(2), 1–17.

Duriez, T., Parezanovic, V., Laurentie, J.C., Fourment, C., Delville, J., Bonnet, J.P., Cordier, L., Noack, B.R., Segond, M., Abel, M.W., Gautier, N., Aider, J.L., Raibaudo, C., Cuvier, C., Stanislas, M., and Brunton, S.L. (2014). Closed-loop control of experimental shear flows using machine learning. In *7th AIAA Flow Control Conference*. American Institute of Aeronautics and Astronautics.

Edwards, C. and Spurgeon, S. (1998). *Sliding Mode Control: Theory And Applications*. CRC Press. Google-Books-ID: uH2RJhIPsiYC.

Efimov, D., Polyakov, A., Fridman, E., Perruquetti, W., and Richard, J.P. (2015). Delay-Dependent Positivity: Application to Interval Observers. URL `https://hal.inria.fr/hal-01140336/document`.

Eldredge, R. and Bons, J. (2004). Active Control of a Separating Boundary Layer with Steady Vortex Generating Jets - Detailed Flow Measurements. In *42nd AIAA Aerospace Sciences Meeting and Exhibit*. American Institute of Aeronautics and Astronautics.

Feingesicht, M., Raibaudo, C., Polyakov, A., Kerherve, F., and Richard, J.P. (2016). A bilinear input-output model with state-dependent delay for separated flow control. URL `https://hal.inria.fr/hal-01298166/document`.

Fernández-Cara, E., Guerrero, S., Imanuvilov, O.Y., and Puel, J.P. (2004). Local exact controllability of the Navier–Stokes system. *Journal de Mathématiques Pures et Appliquées*, 83(12), 1501–1542. doi:10.1016/j.matpur.2004.02.010.

Fridman, E. (2014). *Introduction to Time-Delay Systems: Analysis and Control*. Springer.

Fridman, L., Acosta, P., and Polyakov, A. (2001). Robust eigenvalue assignment for uncertain delay control systems. In *Proc. 3rd IFAC Workshop on Time Delay Systems*, 239–244.

Fridman, L., Strygin, V., and Polyakov, A. (2004). Nonlocal stabilization via delayed relay control rejecting uncertainty in a time delay. *International Journal of Robust and Nonlinear Control*, 14(1), 15–37.

Gauthier, J.P. and Kupka, I. (1992). A separation principle for bilinear systems with dissipative drift. *IEEE transactions on automatic control*, 37(12), 1970–1974.

Ghattas, O. and Bark, J.H. (1997). Optimal Control of Two- and Three-Dimensional Incompressible Navier–Stokes Flows. *Journal of Computational Physics*, 136(2), 231–244. doi:10.1006/jcph.1997.5744.

Hale, J.K. (1971). Caratheodory Conditions. In *Functional Differential Equations*, number 3 in Applied Mathematical Sciences, 30–31. Springer US. DOI: 10.1007/978-1-4615-9968-5_7.

Le Digabel, S. (2011). Algorithm 909: NOMAD: Nonlinear optimization with the MADS algorithm. *ACM Transactions on Mathematical Software (TOMS)*, 37(4), 44.

McManus, K., Legner, H., and Davis, S. (1994). Pulsed vortex generator jets for active control of flow separation. American Institute of Aeronautics and Astronautics. doi:10.2514/6.1994-2218. URL `http://arc.aiaa.org/doi/10.2514/6.1994-2218`.

Olbrot, A. (1976). Control of retarded systems with function space constraints: Necessary optimality conditions. *Control Cybernet*, 5, 5–31.

Polyakov, A. (2012). Minimization of disturbances effects in time delay predictor-based sliding mode control systems. *Journal of the Franklin Institute*, 349(4), 1380–1396. doi:10.1016/j.jfranklin.2011.06.028.

Raibaudo, C., Polyakov, A., Kerherve, F., Richard, J.P., and Stanislas, M. (2015). Experimental open-loop and closed-loop control of a massive separated boundary layer at high Reynolds number.

Raibaudo, C. (2015). *Characterization of the transient of a separated turbulent boundary layer under control and applications to advanced closed-loop controllers*. Ph.D. thesis, Ecole Centrale de Lille.

Raibaudo, C., Kerherve, F., and Stanislas, M. (2013). Characterisation of the transient dynamics of a controlled separated flow using phase averaged PIV. In *International Conference on Instability and Control of Massively Separated Flows – ICOMASEF*.

Raibaudo, C., Polyakov, A., Efimov, D., Kerherve, F., Richard, J.P., and Stanislas, M. (2014). Experimental closed-loop control of a detached boundary layer at high Reynolds number. In *10th EUROMECH Fluid Mechanics Conference*.

Selby, G.V., Lin, J.C., and Howard, F.G. (1992). Control of low-speed turbulent separated flow using jet vortex generators. *Experiments in Fluids*, 12(6), 394–400. doi:10.1007/BF00193886.

Shaqarin, T., Braud, C., Coudert, S., and Stanislas, M. (2013). Open and closed-loop experiments to identify the separated flow dynamics of a thick turbulent boundary layer. *Experiments in Fluids*, 54(2), 1–22. doi:10.1007/s00348-012-1448-4.

Shtessel, Y., Edwards, C., Fridman, L., and Levant, A. (2014). *Sliding mode control and observation*. Springer.

Utkin, V. (1992). *Sliding modes in optimization and control problems*. Springer Verlag, New York.

Volino, R.J. (2003). Separation Control on Low-Pressure Turbine Airfoils Using Synthetic Vortex Generator Jets. 845–859. doi:10.1115/GT2003-38729. URL `http://dx.doi.org/10.1115/GT2003-38729`.

Wachsmuth, D. (2006). *Optimal control of the unsteady Navier-Stokes equations*. Ph.D. thesis, PhD thesis, Technische Universität Berlin.

Yan, X.G., Spurgeon, S.K., and Edwards, C. (2010). Static output feedback sliding mode control for time-varying delay systems with time-delayed nonlinear disturbances. *International Journal of Robust and Nonlinear Control*, 20(7), 777–788.